

Hidden nonlinear $su(2|2)$ superunitary symmetry of $N = 2$ superextended 1D Dirac delta potential problem

Francisco Correa¹, Luis-Miguel Nieto², Mikhail S. Plyushchay¹

¹*Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile*

²*Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071, Valladolid, Spain*

E-mails: fco.correa.s@gmail.com, luismi@metodos.fam.cie.uva.es, mplyushc@lauca.usach.cl

Abstract

We show that the $N = 2$ superextended 1D quantum Dirac delta potential problem is characterized by the hidden nonlinear $su(2|2)$ superunitary symmetry. The unexpected feature of this simple supersymmetric system is that it admits three different \mathbb{Z}_2 -gradings, which produce a separation of 16 integrals of motion into three different sets of 8 bosonic and 8 fermionic operators. These three different graded sets of integrals generate two different nonlinear, deformed forms of $su(2|2)$, in which the Hamiltonian plays a role of a multiplicative central charge. On the ground state, the nonlinear superalgebra is reduced to the two distinct 2D Euclidean analogs of a superextended Poincaré algebra used earlier in the literature for investigation of spontaneous supersymmetry breaking. We indicate that the observed exotic supersymmetric structure with three different \mathbb{Z}_2 -gradings can be useful for the search of hidden symmetries in some other quantum systems, in particular, related to the Lamé equation.

1 Introduction

The Dirac delta potential plays a prominent role in modeling diverse physical systems and phenomena [1, 2, 3, 4]. Recently, it was observed that the simplest *bosonic* 1D quantum Dirac potential problem is characterized by a hidden *nonlocal* $N = 2$ supersymmetry in which a *reflection* plays a role of the grading operator [5]. The corresponding supersymmetry is exact in the case of the attractive potential and is spontaneously broken in the repulsive case.

A priori it is clear that if the system is superextended by introduction of the discrete (spin) degrees of freedom, its explicit supersymmetry should enlarge. The purpose of the present paper is to investigate how the explicit $N = 2$ supersymmetry of the system unifying the attractive and repulsive potential cases is enlarged by the nonlocal hidden supersymmetry.

We show that the $N = 2$ superextended 1D quantum Dirac potential problem is characterized by a nonlinear $su(2|2)$ superunitary symmetry. The unexpected feature of this simple supersymmetric system is that it admits three different possible choices of the grading operator Γ , a usual one given by $\Gamma = \sigma_3$, and two additional given by $\Gamma = R$ and $\Gamma = R\sigma_3$, where R is a reflection operator. These three choices provide us with three different separations of the total set of 16 integrals of motion into 8 bosonic and 8 fermionic operators. The nonlinearity of the superalgebra formed by the integrals of motion is similar to the nonlinearity of the symmetry algebra appearing in the Kepler problem and associated there with a hidden symmetry provided by the Laplace-Runge-Lenz vector [6]. We find that the three choices for the grading operator give rise to the two different forms of nonlinear, deformed $su(2|2)$ supersymmetry, with the Hamiltonian playing the role of the multiplicative central charge of the superalgebra. It is worth to note here that for $\Gamma = R\sigma_3$, the usual supercharges (2.3) play a role of the *bosonic* generators of the superalgebra, whose form in this case is the same as for the choice $\Gamma = \sigma_3$ which identifies (2.3) as *fermionic* integrals.

On the ground state, the two types of superalgebras are reduced to the $2D$ Euclidean versions of the two essentially different cases of a nonstandard superextension of the Poincaré algebra, which was used earlier by Gershun and Tkach in investigation of spontaneous supersymmetry breaking [7].

The paper is organized as follows. In Section 2 we identify the set of integrals of motion of the system, and the form of the nonlinear superalgebra generated by them depending on the choice of the grading operator. In Section 3 we consider the action of the integrals of motion on the energy eigenstates. In Section 4 we identify the nature of the nonlinear supersymmetry of the system. In Section 5 we investigate its reduction on the ground state. Finally, section 6 is devoted to the discussion and concluding remarks.

2 Integrals of motion and grading operators

Consider an $N = 2$ supersymmetric extension of the $1D$ quantum Dirac potential problem described by the Hamiltonian

$$H = p^2 + \beta^2 + 2\beta\delta(x)\sigma_3. \quad (2.1)$$

We choose the units $\hbar = c = 1$, and put the mass of the particle $m = \frac{1}{2}$. In this units, coordinate, momentum, energy and the real parameter $\beta > 0$ are dimensionless. The system is constituted by the two supersymmetric partners corresponding to the attractive (lower component) and the repulsive (upper component) potentials. A usual, the explicit supersymmetry of the system (2.1),

$$\{Q_a, Q_b\} = 2H\delta_{ab}, \quad [Q_a, H] = 0, \quad a = 1, 2, \quad (2.2)$$

is generated by the supercharges

$$Q_1 = p\sigma_1 + \beta\varepsilon(x)\sigma_2, \quad Q_2 = i\sigma_3Q_1. \quad (2.3)$$

Here $\varepsilon(x)$ is the sign function, $\varepsilon(x) = +1$ (-1) for $x > 0$ ($x < 0$) and $\varepsilon(0) = 0$, $\frac{d}{dx}\varepsilon(x) = 2\delta(x)$, which can be treated, e.g., as a limit $\varepsilon(x) = \lim_{\lambda \rightarrow \infty} \tanh \lambda x$, and then $\delta(x) = \lim_{\lambda \rightarrow \infty} \lambda / \cosh^2 \lambda x$. The identities for the Pauli matrices, $\sigma_j \sigma_k = \delta_{jk} + i\epsilon_{jkl}\sigma_l$, give an equivalent representation for (2.3),

$$Q_1 = \sigma_1(p + i\beta\varepsilon(x)\sigma_3), \quad Q_2 = -\sigma_2(p + i\beta\varepsilon(x)\sigma_3). \quad (2.4)$$

For the $N = 2$ superalgebra (2.2), a \mathbb{Z}_2 -grading operator Γ ,

$$\Gamma^2 = 1, \quad [\Gamma, H] = 0, \quad \{\Gamma, Q_a\} = 0, \quad (2.5)$$

is identified usually with the diagonal σ -matrix,

$$\Gamma = \sigma_3. \quad (2.6)$$

This choice is not a unique, however, and the identification of the grading operator with the reflection,

$$\Gamma = R, \quad (2.7)$$

$R\psi(x) = \psi(-x)$, is also consistent here with relations (2.5).

In accordance with [5], the system has also the integrals of motion

$$\tilde{Q}_1 = p + i\beta\varepsilon(x)R\sigma_3, \quad \tilde{Q}_2 = iR\tilde{Q}_1, \quad (2.8)$$

which, like R , are nonlocal operators. Taking into account the identity $\delta(x)R = \delta(x)$, one finds that like usual supercharges (2.3), they are square root of the Hamiltonian, $\tilde{Q}_1^2 = \tilde{Q}_2^2 = H$. Integrals

(2.8) are the \mathbb{Z}_2 -odd operators (supercharges) for the choice (2.7), but have to be treated as \mathbb{Z}_2 -even operators for (2.6). In the first case, they generate another copy of the $N = 2$ superalgebra.

There is a third possibility for identification of the grading operator,

$$\Gamma = R\sigma_3. \quad (2.9)$$

With respect to this grading, operators \tilde{Q}_a are odd, while Q_a are even.

Let us see now what is the whole set of even and odd integrals of motion of the system for each of the three different grading operators we have just specified.

2.1 Grading $\Gamma = R$

Since with respect to (2.7) both sets of integrals (2.3) and (2.8) are odd operators, let us fix this grading and analyze further the symmetries of the system. With respect to the other two gradings, one of the two sets of integrals (2.3) and (2.8) should be treated as odd, and the other as even. These alternative choices of the grading will be considered separately, but we shall see finally that in all the three cases the structure of the resulting complete supersymmetry is the same, modulo the Hamiltonian.

For the grading (2.7), the anticommutators between the integrals (2.3) and (2.8) should be calculated. This gives

$$\{Q_a, \tilde{Q}_1\} = 2S_a, \quad \{Q_a, \tilde{Q}_2\} = 0, \quad a = 1, 2, \quad (2.10)$$

where

$$S_1 = \sigma_1 H - \beta \varepsilon(x) Q_2 (1 + R), \quad S_2 = i\sigma_3 S_1. \quad (2.11)$$

Hermitian operators (2.11) should be treated as new even integrals of motion. Then, the process has to be continued: it is necessary to calculate the commutators of these even integrals of motion between themselves and with the odd integrals, etc. As a result, we obtain the complete list of odd (F_1, \dots, F_8) and even ($H, R, \Sigma_1, \Sigma_2, B_1, \dots, B_4$) integrals of motion, which can be represented in terms of the basic ‘building block operators’ $Q_1, \tilde{Q}_1, S_1, \sigma_3$ and R . They are shown in Table 1.

Fermionic integrals	$F_1 = Q_1$	$F_2 = R\sigma_3 Q_1$	$F_3 = Q_2 = i\sigma_3 Q_1$	$F_4 = iRQ_1$
	$F_5 = iR\tilde{Q}_1 = \tilde{Q}_2$	$F_6 = iR\sigma_3 \tilde{Q}_1$	$F_7 = \sigma_3 \tilde{Q}_1$	$F_8 = \tilde{Q}_1$
Bosonic integrals	H	$\Gamma = R$	$\Sigma_1 = \sigma_3$	$\Sigma_2 = R\sigma_3$
	$B_1 = Q_1 \tilde{Q}_1 = S_1$	$B_2 = i\sigma_3 S_1 = S_2$	$B_3 = RS_1$	$B_4 = iR\sigma_3 S_1$

Table 1: Integrals of motion, grading $\Gamma = R$

The anticommutation relations between the fermionic integrals are presented in Table 2.

The commutation relations between bosonic integrals and between bosonic and fermionic operators are listed in Tables 3 and 4.

Here we have introduced the parameter λ to include also the other two cases of \mathbb{Z}_2 -grading. In the present case of $\Gamma = R$, $\lambda = 0$. Due to the special choice of the indices in the definition of fermionic operators F and bosonic operators B , the tables of (anti)commutation relations have a simple block form. The Hamiltonian H commutes with all the fermionic and bosonic operators. The grading operator $\Gamma = R$ commutes with all the bosonic integrals.

2.2 Grading $\Gamma = \sigma_3$

For the choice of the \mathbb{Z}_2 -grading operator $\Gamma = \sigma_3$, as we noted, the integrals (2.3) are, again, the fermionic operators, but the integrals (2.8) are even operators. As a result, now the operators S_a

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
F_1	$2H$	$2\Sigma_2 H$	0	0	0	$2B_4 H^\lambda$	0	$2B_1 H^\lambda$
F_2	$2\Sigma_2 H$	$2H$	0	0	$-2B_2 H^\lambda$	0	$-2B_3 H^\lambda$	0
F_3	0	0	$2H$	$2\Sigma_2 H$	0	$-2B_3 H^\lambda$	0	$2B_2 H^\lambda$
F_4	0	0	$2\Sigma_2 H$	$2H$	$2B_1 H^\lambda$	0	$-2B_4 H^\lambda$	0
F_5	0	$-2B_2 H^\lambda$	0	$2B_1 H^\lambda$	$2H^{1+\lambda}$	$2\Sigma_1 H^{1+\lambda}$	0	0
F_6	$2B_4 H^\lambda$	0	$-2B_3 H^\lambda$	0	$2\Sigma_1 H^{1+\lambda}$	$2H^{1+\lambda}$	0	0
F_7	0	$-2B_3 H^\lambda$	0	$-2B_4 H^\lambda$	0	0	$2H^{1+\lambda}$	$2\Sigma_1 H^{1+\lambda}$
F_8	$2B_1 H^\lambda$	0	$2B_2 H^\lambda$	0	0	0	$2\Sigma_1 H^{1+\lambda}$	$2H^{1+\lambda}$

Table 2: Fermion-fermion anticommutation relations

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
Γ	$-2iF_4$	$-2iF_3$	$2iF_2$	$2iF_1$	$2iF_8$	$2iF_7$	$-2iF_6$	$-2iF_5$
Σ_1	$-2iF_3$	$-2iF_4$	$2iF_1$	$2iF_2$	0	0	0	0
Σ_2	0	0	0	0	$2iF_7$	$2iF_8$	$-2iF_5$	$-2iF_6$
B_1	0	$2iF_6 H^{1-\lambda}$	$-2iF_7 H^{1-\lambda}$	0	0	$-2iF_2 H$	$2iF_3 H$	0
B_2	$2iF_7 H^{1-\lambda}$	0	0	$2iF_6 H^{1-\lambda}$	0	$-2iF_4 H$	$-2iF_1 H$	0
B_3	$-2iF_5 H^{1-\lambda}$	0	0	$2iF_8 H^{1-\lambda}$	$2iF_1 H$	0	0	$-2iF_4 H$
B_4	0	$-2iF_8 H^{1-\lambda}$	$-2iF_5 H^{1-\lambda}$	0	$2iF_3 H$	0	0	$2iF_2 H$

Table 3: Boson-fermion commutation relations

are odd operators appearing from the commutation relations of (2.3) with (2.8). The identification of the fermionic and bosonic integrals of motion for this case is presented in Table 5.

These operators satisfy the (anti)commutation relations of the same form as in the previous case, but now with $\lambda = 1$.

2.3 Grading $\Gamma = R\sigma_3$

The choice $\Gamma = R\sigma_3$ is similar to the choice (2.6). The difference with the previous case is that now the integrals (2.3) are identified as even operators, while the integrals (2.8) have a nature of odd operators. The integrals S_a , together with two other integrals related to them, play again here the role of odd operators. The identification of the fermionic and bosonic operators is given in Table 6.

These operators satisfy the (anti)commutation relations exactly of the same form as for grading (2.6), i.e. the parameter λ is assigned here the value $\lambda = 1$.

2.4 Identification of operators

The identification of the fermionic and bosonic operators has been realized by us in a special way which guarantees the same form (modulo H) of the (anti)commutation relations between them for the three possible choices of the \mathbb{Z}_2 -grading operator. The identification we use corresponds to the following simple procedure.

Step 1. Choose the grading operator Γ from the set of the three Hermitian operators

$$\{R, \quad \sigma_3, \quad R\sigma_3\}. \quad (2.12)$$

	Σ_1	Σ_2	B_1	B_2	B_3	B_4
Σ_1	0	0	$-2iB_2$	$2iB_1$	$-2iB_4$	$2iB_3$
Σ_2	0	0	$-2iB_4$	$2iB_3$	$-2iB_2$	$2iB_1$
B_1	$2iB_2$	$2iB_4$	0	$-2i\Sigma_1 H^{2-\lambda}$	0	$-2i\Sigma_2 H^{2-\lambda}$
B_2	$-2iB_1$	$-2iB_3$	$2i\Sigma_1 H^{2-\lambda}$	0	$2i\Sigma_2 H^{2-\lambda}$	0
B_3	$2iB_4$	$2iB_2$	0	$-2i\Sigma_2 H^{2-\lambda}$	0	$-2i\Sigma_1 H^{2-\lambda}$
B_4	$-2iB_3$	$-2iB_1$	$2i\Sigma_2 H^{2-\lambda}$	0	$2i\Sigma_1 H^{2-\lambda}$	0

Table 4: Boson-boson commutation relations

Fermionic integrals	$F_1 = Q_1$	$F_2 = -R\sigma_3 Q_1$	$F_3 = -iRQ_1$	$F_4 = Q_2 = i\sigma_3 Q_1$
	$F_5 = RS_1$	$F_6 = -S_1$	$F_7 = i\sigma_3 S_1$	$F_8 = -iR\sigma_3 S_1$
Bosonic integrals	H	$\Sigma_1 = -R$	$\Gamma = \sigma_3$	$\Sigma_2 = -R\sigma_3$
	$B_1 = -iR\sigma_3 \tilde{Q}_1$	$B_2 = -\sigma_3 \tilde{Q}_1$	$B_3 = -iR\tilde{Q}_1$	$B_4 = -\tilde{Q}_1$

Table 5: Integrals of motion, grading $\Gamma = \sigma_3$

Fermionic integrals	$F_1 = \tilde{Q}_1$	$F_2 = -\sigma_3 \tilde{Q}_1$	$F_3 = -iR\tilde{Q}_1$	$F_4 = iR\sigma_3 \tilde{Q}_1$
	$F_5 = RS_1$	$F_6 = -S_1$	$F_7 = iR\sigma_3 S_1$	$F_8 = -i\sigma_3 S_1$
Bosonic integrals	H	$\Sigma_1 = -R$	$\Sigma_2 = -\sigma_3$	$\Gamma = R\sigma_3$
	$B_1 = -i\sigma_3 Q_1$	$B_2 = -R\sigma_3 Q_1$	$B_3 = -iRQ_1$	$B_4 = -Q_1$

Table 6: Integrals of motion, grading $\Gamma = R\sigma_3$

Step 2. Select any Hermitian fermionic operator F_1 , $\{F_1, \Gamma\} = 0$, from the set of the eight odd integrals with the following properties: $\{F_1, F_1\} = 2H$ and $[\Sigma_2, F_1] = 0$. Here we denote by $\Sigma_2 \neq \Gamma$ a bosonic operator from the set (2.12) which commutes with F_1 . The third operator from (2.12) is denoted by Σ_1 . The operators Σ_1 and Σ_2 are defined up to a sign.

Step 3. With the commutation relations from Table 3 of F_1 with Γ and Σ_1 , the fermionic operators F_2 , F_3 and F_4 are fixed.

Step 4. Choose any other fermionic operator F_5 that satisfies $\{F_1, F_5\} = 0$ and $\{F_5, F_5\} = 2H^{1+\lambda}$.

Step 5. Repeat Step 3, but changing $F_1 \rightarrow F_5$ and $\Sigma_1 \rightarrow \Sigma_2$ to obtain F_6 , F_7 and F_8 .

Step 6. With the anticommutators of the Tables 2 we identify the bosonic operators B_1 , B_2 , B_3 and B_4 .

3 The action of the integrals of motion on the energy eigenstates

We have three types of the ‘basic’ integrals of motion, Q_a , \tilde{Q}_a , S_a , in terms of which other integrals are obtained by multiplication with the even integrals R and σ_3 . Here we will trace out the difference between the nature of these operators by applying them to the physical states which we choose finally to be the eigenfunctions of the operators H , R and σ_3 .

In the *repulsive* case (the upper component subsystem), the Hamiltonian is $H = p^2 + 2\beta\delta(x) + \beta^2$, and the eigenfunctions with energy $E = k^2 + \beta^2 > \beta^2$, $k > 0$, are given by

$$\psi_k^+(x) = (e^{ikx} + re^{-ikx})\Theta(-x) + te^{ikx}\Theta(x), \quad \psi_k^-(x) = \psi_k^+(-x), \quad (3.1)$$

where $\Theta(x)$ is the Heaviside step function. These wavefunctions correspond to scattering states with the plane waves incoming, respectively, from $-\infty$ and $+\infty$, with reflection and transmission coefficients given by [5]

$$r = \frac{\beta}{ik - \beta}, \quad t = \frac{ik}{ik - \beta}, \quad k = \sqrt{E - \beta^2}. \quad (3.2)$$

Let us construct from these states the eigenfunctions of a fixed parity,

$$\psi_{k,+}^{(+)}(x) = \frac{(ik - \beta)}{2i}(\psi_k^+ + \psi_k^-) = k \cos kx + \beta\varepsilon(x) \sin kx, \quad (3.3)$$

$$\psi_{k,-}^{(+)}(x) = \frac{1}{2i}(\psi_k^+ - \psi_k^-) = \sin kx, \quad (3.4)$$

$$R\psi_{k,\pm}^{(+)}(x) = \pm\psi_{k,\pm}^{(+)}(x).$$

The *attractive* (lower component) case is obtained by the change $\beta \rightarrow -\beta$ in all the previous formulae, $H = p^2 - 2\beta\delta(x) + \beta^2$,

$$\psi_{k,+}^{(-)}(x) = k \cos kx - \beta\varepsilon(x) \sin kx, \quad \psi_{k,-}^{(-)}(x) = \sin kx. \quad (3.5)$$

In addition to the scattering states, this subsystem has a unique bound state described by a normalized wave function,

$$\psi_0^{(-)}(x) = \sqrt{\beta}e^{-\beta|x|}. \quad (3.6)$$

Due to a special value of the constant term in the Hamiltonian, the energy of this bound state is equal to zero.

Summarizing, for the system (2.1) we have the scattering eigenstates,

$$\Psi_{k,+}^{(+)}(x) = \begin{pmatrix} k \cos kx + \beta\varepsilon(x) \sin kx \\ 0 \end{pmatrix}, \quad \Psi_{k,-}^{(+)}(x) = \sqrt{k^2 + \beta^2} \begin{pmatrix} \sin kx \\ 0 \end{pmatrix}, \quad (3.7)$$

$$\Psi_{k,+}^{(-)}(x) = \begin{pmatrix} 0 \\ k \cos kx - \beta\varepsilon(x) \sin kx \end{pmatrix}, \quad \Psi_{k,-}^{(-)}(x) = \sqrt{k^2 + \beta^2} \begin{pmatrix} 0 \\ \sin kx \end{pmatrix}, \quad (3.8)$$

and the bound state of zero energy,

$$\Psi_0^{(-)}(x) = \begin{pmatrix} 0 \\ \sqrt{\beta}e^{-\beta|x|} \end{pmatrix}. \quad (3.9)$$

We have introduced here a convenient normalization to simplify the relations that follow.

The ordinary supercharges (2.3) change the *parity* and the *upper-lower* subspaces of the eigenfunctions,

$$Q_1\Psi_{\pm}^{(\pm)} = \pm i\sqrt{k^2 + \beta^2}\Psi_{\mp}^{(\mp)}, \quad Q_1\Psi_{\mp}^{(\pm)} = \mp i\sqrt{k^2 + \beta^2}\Psi_{\pm}^{(\mp)}, \quad (3.10)$$

$$Q_2\Psi_{\pm}^{(\pm)} = \sqrt{k^2 + \beta^2}\Psi_{\mp}^{(\mp)}, \quad Q_2\Psi_{\mp}^{(\pm)} = -\sqrt{k^2 + \beta^2}\Psi_{\pm}^{(\mp)}, \quad (3.11)$$

where for simplicity we have omitted the index k . The supercharges of the *hidden* supersymmetry (2.8) change only the *parity*,

$$\tilde{Q}_1\Psi_{\pm}^{(\alpha)} = \pm i\sqrt{k^2 + \beta^2}\Psi_{\mp}^{(\alpha)}, \quad \tilde{Q}_2\Psi_{\pm}^{(\alpha)} = \sqrt{k^2 + \beta^2}\Psi_{\mp}^{(\alpha)}, \quad \alpha = +, -, \quad (3.12)$$

The S_a operators change only the *upper-lower subspaces* of eigenfunctions,

$$S_1 \Psi_\alpha^{(\pm)} = (k^2 + \beta^2) \Psi_\alpha^{(\mp)}, \quad S_2 \Psi_\alpha^{(\pm)} = \pm i(k^2 + \beta^2) \Psi_\alpha^{(\mp)}, \quad \alpha = +, -. \quad (3.13)$$

The unique bound state is annihilated by the integrals of motion Q_a , \tilde{Q}_a and S_a ,

$$Q_a \Psi_0^{(-)} = 0, \quad \tilde{Q}_a \Psi_0^{(-)} = 0, \quad S_a \Psi_0^{(-)} = 0. \quad (3.14)$$

As a consequence, it is annihilated by all the odd integrals of motion F_1, \dots, F_8 and all the even integrals B_1, \dots, B_4 (as well as by H).

4 Identification of supersymmetry, generic case

To clarify the nature of the supersymmetry generated by the complete set of eight fermionic and eight bosonic integrals of motion, we define the following linear combinations of the even operators,

$$P_1^{(\pm)} = \frac{1}{4} (B_1 \pm B_3) = \frac{1}{2} B_1 \Pi_\pm, \quad P_2^{(\pm)} = -\frac{1}{4} (B_2 \pm B_4) = -\frac{1}{2} B_2 \Pi_\pm, \quad (4.1)$$

$$J_3^{(\pm)} = \frac{1}{4} (\Sigma_1 \pm \Sigma_2) = \frac{1}{2} \Sigma_1 \Pi_\pm, \quad (4.2)$$

where $\Pi_\pm = \frac{1}{2}(1 \pm \Gamma)$ are the projectors. These operators satisfy the relations

$$[P_1^{(\pm)}, P_2^{(\pm)}] = i J_3^{(\pm)} H^{2-\lambda}, \quad (4.3)$$

$$[J_3^{(\pm)}, P_a^{(\pm)}] = i \epsilon_{ab} P_b^{(\pm)}, \quad a, b = 1, 2, \quad (4.4)$$

$$[P_a^{(+)}, P_b^{(-)}] = [J_3^{(+)}, P_a^{(-)}] = [J_3^{(-)}, P_a^{(+)}] = [J_3^{(+)}, J_3^{(-)}] = 0. \quad (4.5)$$

The commutation relations (4.3), (4.4) correspond to a direct sum of two deformed $su(2)$ algebras. In particular, relations (4.3) are reminiscent of the commutation relations satisfied by the components of the Laplace-Runge-Lenz vector in the quantum Kepler problem. Being reduced to the energy eigensubspace of nonzero eigenvalue $E > \beta^2 > 0$, the rescaled operators $J_a^{(\pm)} = P_a^{(\pm)} / E^{1-\frac{\lambda}{2}}$ and $J_3^{(\pm)}$ generate the Lie algebra $su(2) \oplus su(2)$ and satisfy the relations $J_i^{(+)} J_i^{(+)} = \frac{3}{4} \Pi_+$, $J_i^{(-)} J_i^{(-)} = \frac{3}{4} \Pi_-$, where the summation in $i = 1, 2, 3$ is assumed. These relations mean that the two common eigenstates of the Hamiltonian with energy $E > \beta^2 > 0$ and of the grading operator Γ with eigenvalue $+1$ carry spin-1/2 representation for $J_i^{(+)}$, and are the spin-0 states for $J_i^{(-)}$. The states with $\Gamma = -1$ carry spin one-half representation for $J_i^{(-)}$, and are of spin zero for $J_i^{(+)}$. Fermionic integrals mutually transform the states from these eigenspaces of the grading operator. In accordance with the total number of independent fermionic generators, the energy subspace with $E > \beta^2 > 0$ carries an irreducible representation of the $su(2|2)$ superunitary symmetry, which is a supersymmetric extension of the bosonic symmetry $u(1) \oplus su(2) \oplus su(2)$, where the $u(1)$ subalgebra is generated by the grading operator, see Ref. [8]. Having in mind that the Hamiltonian appears in a generic form of the superalgebra as a multiplicative central charge, we conclude that the system possesses the nonlinear $su(2|2)$ superunitary symmetry in the sense of Refs. [6, 9, 10].

In order to understand better the identified symmetry at the algebraic level, let us define the following linear complex combinations of the even operators (4.1), $P_\pm^{(\pm)} = P_1^{(\pm)} \pm i P_2^{(\pm)}$, and of the odd operators,

$$\mathcal{S}_1 = \frac{1}{2} (F_1 + i F_3), \quad \mathcal{S}_2 = \frac{1}{2} (F_2 + i F_4), \quad \mathcal{Q}_1 = \frac{1}{2} (F_7 + i F_5), \quad \mathcal{Q}_2 = \frac{1}{2} (F_8 + i F_6), \quad (4.6)$$

	\mathcal{S}_1	$\bar{\mathcal{S}}_1$	\mathcal{S}_2	$\bar{\mathcal{S}}_2$	\mathcal{Q}_1	$\bar{\mathcal{Q}}_1$	\mathcal{Q}_2	$\bar{\mathcal{Q}}_2$
\mathcal{S}_1	0	H	0	$\Sigma_2 H$	0	0	$2P_-^{(+)} H^\lambda$	$2P_-^{(-)} H^\lambda$
$\bar{\mathcal{S}}_1$	H	0	$\Sigma_2 H$	0	0	0	$2P_+^{(-)} H^\lambda$	$2P_+^{(+)} H^\lambda$
\mathcal{S}_2	0	$\Sigma_2 H$	0	H	$-2P_-^{(+)} H^\lambda$	$2P_-^{(-)} H^\lambda$	0	0
$\bar{\mathcal{S}}_2$	$\Sigma_2 H$	0	H	0	$2P_+^{(-)} H^\lambda$	$-2P_+^{(+)} H^\lambda$	0	0
\mathcal{Q}_1	0	0	$-2P_-^{(+)} H^\lambda$	$2P_+^{(-)} H^\lambda$	0	$H^{1+\lambda}$	0	$\Sigma_1 H^{1+\lambda}$
$\bar{\mathcal{Q}}_1$	0	0	$2P_-^{(-)} H^\lambda$	$-2P_+^{(+)} H^\lambda$	$H^{1+\lambda}$	0	$\Sigma_1 H^{1+\lambda}$	0
\mathcal{Q}_2	$2P_-^{(+)} H^\lambda$	$2P_+^{(-)} H^\lambda$	0	0	0	$\Sigma_1 H^{1+\lambda}$	0	$H^{1+\lambda}$
$\bar{\mathcal{Q}}_2$	$2P_-^{(-)} H^\lambda$	$2P_+^{(+)} H^\lambda$	0	0	$\Sigma_1 H^{1+\lambda}$	0	$H^{1+\lambda}$	0

Table 7: Fermion-fermion anticommutation relations. Here $\Sigma_{1(2)} = 2 \left(J_3^{(+)} + (-) J_3^{(-)} \right)$

	\mathcal{S}_1	$\bar{\mathcal{S}}_1$	\mathcal{S}_2	$\bar{\mathcal{S}}_2$	\mathcal{Q}_1	$\bar{\mathcal{Q}}_1$	\mathcal{Q}_2	$\bar{\mathcal{Q}}_2$
Γ	$-2\mathcal{S}_2$	$2\bar{\mathcal{S}}_2$	$-2\mathcal{S}_1$	$2\bar{\mathcal{S}}_1$	$-2\mathcal{Q}_2$	$2\bar{\mathcal{Q}}_2$	$-2\mathcal{Q}_1$	$2\bar{\mathcal{Q}}_1$
$J_3^{(+)}$	$-\frac{1}{2}\mathcal{S}_1$	$\frac{1}{2}\bar{\mathcal{S}}_1$	$-\frac{1}{2}\mathcal{S}_2$	$\frac{1}{2}\bar{\mathcal{S}}_2$	$-\frac{1}{2}\mathcal{Q}_1$	$\frac{1}{2}\bar{\mathcal{Q}}_1$	$-\frac{1}{2}\mathcal{Q}_2$	$\frac{1}{2}\bar{\mathcal{Q}}_2$
$J_3^{(-)}$	$-\frac{1}{2}\mathcal{S}_1$	$\frac{1}{2}\bar{\mathcal{S}}_1$	$-\frac{1}{2}\mathcal{S}_2$	$\frac{1}{2}\bar{\mathcal{S}}_2$	$\frac{1}{2}\mathcal{Q}_1$	$-\frac{1}{2}\bar{\mathcal{Q}}_1$	$\frac{1}{2}\mathcal{Q}_2$	$-\frac{1}{2}\bar{\mathcal{Q}}_2$
$P_-^{(+)}$	0	$-\mathcal{Q}_1 H^{1-\lambda}$	0	$\mathcal{Q}_2 H^{1-\lambda}$	0	$\mathcal{S}_1 H$	0	$-\mathcal{S}_2 H$
$P_-^{(-)}$	0	$-\bar{\mathcal{Q}}_1 H^{1-\lambda}$	0	$-\bar{\mathcal{Q}}_2 H^{1-\lambda}$	$\mathcal{S}_1 H$	0	$\mathcal{S}_2 H$	0
$P_+^{(+)}$	$\bar{\mathcal{Q}}_1 H^{1-\lambda}$	0	$-\bar{\mathcal{Q}}_2 H^{1-\lambda}$	0	$-\bar{\mathcal{S}}_1 H$	0	$\bar{\mathcal{S}}_2 H$	0
$P_+^{(-)}$	$\mathcal{Q}_1 H^{1-\lambda}$	0	$\mathcal{Q}_2 H^{1-\lambda}$	0	0	$-\bar{\mathcal{S}}_1 H$	0	$-\bar{\mathcal{S}}_2 H$

Table 8: Boson-fermion commutation relations

and denote $\bar{\mathcal{S}}_{1,2} = \mathcal{S}_{1,2}^\dagger$, $\bar{\mathcal{Q}}_{1,2} = \mathcal{Q}_{1,2}^\dagger$. In terms of these complex combinations, the (anti)commutation relations including fermionic integrals are presented in Tables 7 and 8.

From the (anti)commutation relations it follows that the operators $J_3^{(+)}$, and $P_\pm^{(+)}$ act invariantly (in the sense of commutator) on every of the spin one-half multiplets $(\bar{\mathcal{Q}}_1, \mathcal{S}_1)$, $(\bar{\mathcal{S}}_1, \mathcal{Q}_1)$, $(\mathcal{Q}_2, \mathcal{S}_2)$, $(\bar{\mathcal{S}}_2, \mathcal{Q}_2)$, where the first and second components of every multiplet are the eigenstates of $J_3^{(+)}$ with eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$. Analogous multiplets for the bosonic operators $J_3^{(-)}$, and $P_\pm^{(-)}$ are $(\mathcal{Q}_1, \mathcal{S}_1)$, $(\bar{\mathcal{S}}_1, \bar{\mathcal{Q}}_1)$, $(\mathcal{Q}_2, \mathcal{S}_2)$ and $(\bar{\mathcal{S}}_2, \bar{\mathcal{Q}}_2)$. The eigenstates of the $u(1)$ generator $\frac{1}{2}\Gamma$ are the linear combinations of (4.6) and of the Hermitian conjugate operators: the operators $(\mathcal{S}_1 - \mathcal{S}_2)$, $(\bar{\mathcal{S}}_1 + \bar{\mathcal{S}}_2)$, $(\mathcal{Q}_1 - \mathcal{Q}_2)$ and $(\bar{\mathcal{Q}}_1 + \bar{\mathcal{Q}}_2)$ are the eigenstates of the eigenvalue $+1$, while $(\mathcal{S}_1 + \mathcal{S}_2)$, $(\bar{\mathcal{S}}_1 - \bar{\mathcal{S}}_2)$, $(\mathcal{Q}_1 + \mathcal{Q}_2)$ and $(\bar{\mathcal{Q}}_1 - \bar{\mathcal{Q}}_2)$ are the eigenstates of the eigenvalue -1 . The anticommutators of the odd integrals produce linear combinations of H and the generators of the two chiral $su(2)$ algebras modulo H .

5 Supersymmetry reduced on the ground state

As we have seen, the three possible gradings result in only two forms of the supersymmetry algebra given by the (anti)commutation relations with $\lambda = 0$ for $\Gamma = R$, and with $\lambda = 1$ for $\Gamma = \sigma_3$ and $\Gamma = R\sigma_3$. The reduction of these superalgebras on the subspace of scattering states with fixed energy $E > \beta^2 > 0$ effectively results, for both values of λ , in the same superextension of the $u(1) \oplus su(2) \oplus su(2)$ Lie algebra. However, the essential difference between the two forms reveals itself on the ground state of zero energy.

Indeed, with $H = 0$, the bosonic part of the superalgebra is reduced to the algebra $u(1) \oplus e(2) \oplus e(2)$, where the first term corresponds to the integral Γ , and other two correspond to the two copies of the 2D Euclidean algebras generated by the rotation operator $J_3^{(+)}$, the translation generators $P_a^{(+)}$, $[P_1^{(+)}, P_2^{(+)}] = 0$, and by its analogs for the $\Gamma = -1$ subspace.

For $\lambda = 1$, in accordance with Table 7, all the fermionic operators anticommute, and in this case we have also the commutation relations of the form $[P, Q] = 0$, $[P, S] \sim Q$, where we do not display the indices and do not distinguish fermionic operators from the Hermitian conjugate.

On the other hand, for $\lambda = 0$, the fermionic integrals commute with the translation generators, $[P, S] = [P, Q] = 0$, and anticommute between themselves in the following way: $\{S, S\} = \{Q, Q\} = 0$, $\{S, Q\} \sim P$. Therefore, the difference between the grading $\Gamma = R$ on one hand, and the gradings $\Gamma = \sigma_3$ and $\Gamma = R\sigma_3$ on the other hand, reveals itself on the ground state.

It is interesting to note that the supersymmetry of the form similar to the present supersymmetry reduced on the ground state was analyzed by Gershun and Tkach [7] in the context of spontaneous supersymmetry breaking in 3+1 dimensions. The superalgebra used by them contains two parameters, a and b , (see Eq. (1) from [7]). The cases $(a = 0, b \neq 0)$ and $(a \neq 0, b = 0)$ correspond, respectively, to our cases $\lambda = 0$ and $\lambda = 1$.

6 Concluding remarks and outlook

Although the system we have studied is simple, the existence of nontrivial (nonlocal) integrals of motion reveals a new supersymmetric structure. A part of the nontrivial integrals of motion of the $N = 2$ superextended system comes from the hidden supersymmetry of the corresponding bosonic (spinless) quantum mechanical Dirac delta potential problem, while other conserved quantities appear naturally with the addition of fermionic degrees of freedom. Combining both supersymmetries, the usual one generated by (2.3), and the hidden supersymmetry generated by (2.8), we find new, a priori unexpected nontrivial (nonlocal) integrals of motion (2.11). As a result, we enrich the knowledge on the system with a new property: its supersymmetric structure admits three different grading operators. With respect to these three different \mathbb{Z}_2 -gradings, every of the three sets of nontrivial integrals (2.3), (2.8) and (2.11) (and other associated integrals obtained from them by multiplication with grading operators) is identified once as a set of bosonic generators, while two other sets are identified in this case as fermionic operators.

Hidden supersymmetry exists also in a bosonic (spinless) Pöschl-Teller (PT) quantum problem for integer values of the parameter of the system (for the details see ref. [5]), the appropriate limit of which corresponds to the Dirac potential problem. However, in PT problem, hidden $N = 2$ supersymmetry is nonlinear [9, 10], of the order defined by the value of the parameter. Therefore, one should expect that such a hidden nonlinear supersymmetry has also to reveal itself in the simplest $N = 2$ superextended version of the PT system. However, due to the nonlinear character of the hidden supersymmetry, the resulting complete supersymmetry has to have a more rich nature in comparison with the Dirac delta potential case. Moreover, since the PT problem at those special values of the parameter (being reflectionless) carries some characteristics of a free particle, and in a certain limit produces a particle model of conformal mechanics [11], it would be interesting to investigate the PT system from the viewpoint of possible hidden superconformal symmetry.

Another system that reveals hidden nonlinear supersymmetry is the quantum periodic Lamé problem [12], which reproduces the hidden supersymmetry of the PT system in a certain limit. Following the line presented in this work, we also are going to investigate the question on hidden supersymmetry in the superextended Lamé system, which has some remarkable properties in the context of supersymmetry [13].

Let us note that the observed exotic supersymmetric structure with three different \mathbb{Z}_2 -gradings can be effective in the search of hidden symmetries in some other quantum systems. In particular,

a general knowledge of this structure is helpful for identification of the unknown hidden symmetries in the broad class of the associated Lamé systems [14], and in solving them. The key point there is that to every Lamé system two different algebrisation schemes are related [15], with which two different sets of the non-diagonal integrals of the type (2.3) and (2.11) can be naturally associated. The detailed study of Lamé and Pöschl-Teller systems in the light of the revealed here nontrivial supersymmetric structure will be presented elsewhere [16].

Since the reduction of the observed hidden nonlinear $su(2|2)$ supersymmetry of the $N = 2$ superextended Dirac delta potential problem to the ground state of zero energy produces the 2D Euclidean analogs of two particular cases of a special superextension of the Poincaré symmetry [7], it would be interesting to investigate the origin of this latter symmetry. One could expect that it should appear in a special (contraction) limit of some superextension of the de Sitter symmetry.

Finally, it would be interesting to investigate the question on existence of the hidden supersymmetry in the systems of many particles, in particular, with Dirac delta interactions, as well as in nonlinear integrable systems.

Acknowledgements. We are grateful to Dmitri Sorokin for valuable communications. The work has been supported partially by the FONDECYT Projects 1050001 and 7070024, by CONICYT, by Spanish Ministerio de Educación y Ciencia (Project MTM2005-09183) and Junta de Castilla y León (Excellence Project VA013C05). FC and MP thank Department of Theoretical Physics of Valladolid University for hospitality.

References

- [1] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967); Phys. Rev. **168**, 1920 (1968); C. N. Yang and C. P. Yang, J. Math. Phys. **10** (1969) 1115.
- [2] C. R. Hagen, Int. J. Mod. Phys. A **6** (1991) 3119.
- [3] C. Manuel and R. Tarrach, Phys. Lett. B **328**, 113 (1994); S. K. Adhikari and T. Frederico, Phys. Rev. Lett. **74**, 4572 (1995).
- [4] R. Jackiw, in M. A. B. Bég Memorial Volume, ed. by A. Ali and P. Hoodbhoy (World Scientific, Singapore, 1991).
- [5] F. Correa and M. S. Plyushchay, Ann. Phys. **322**, 2493 (2007).
- [6] J. de Boer, F. Harmsze and T. Tjin, Phys. Rept. **272**, 139 (1996).
- [7] V. D. Gershun and V. I. Tkach, Ukr. Fiz. Zh. **25**, 36 (1980).
- [8] D. Sen, Phys. Rev. D **41**, 667 (1990).
- [9] A. A. Andrianov, M. V. Ioffe and V. P. Spiridonov, Phys. Lett. A **174**, 273 (1993).
- [10] M. S. Plyushchay, Ann. Phys. **245**, 339 (1996); Int. J. Mod. Phys. A **15**, 3679 (2000).
- [11] V. de Alfaro, S. Fubini and G. Furlan, Nuovo Cim. A **34** (1976) 569; V. P. Akulov and A. I. Pashnev, Theor. Math. Phys. **56** (1983) 862 [Teor. Mat. Fiz. **56** (1983) 344]; S. Fubini and E. Rabinovici, Nucl. Phys. B **245** (1984) 17; A. Anabalon and M. S. Plyushchay, Phys. Lett. B **572** (2003) 202.
- [12] F. Correa, L.-M. Nieto and M. S. Plyushchay, Phys. Lett. B **644**, 94 (2007).
- [13] G. V. Dunne and J. Feinberg, Phys. Rev. D **57**, 1271 (1998).

- [14] W. Magnus and S. Winkler, Hills Equation (Wiley, New York, 1966); A. Khare and U. Sukhatme, J. Math. Phys. **40**, 5473 (1999); D. J. Fernandez C. and A. Ganguly, Phys. Lett. A **338**, 203 (2005); A. Ganguly, M. V. Ioffe and L. M. Nieto, J. Phys. A **39**, 14659 (2006).
- [15] A. V. Turbiner, Commun. Math. Phys. **118**, 467 (1988); A. Ganguly, Mod. Phys. Lett. A **15**, 1923 (2000).
- [16] F. Correa, V. Jakubsky, L.-M. Nieto and M. S. Plyushchay, in preparation.